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LOCATION OF ZEROS OF CERTAIN POLYNOMIALS IN ANNULAR REGIONS

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Abstract: The famous Eneström-Kakeya Theorem states that a polynomial $P(z) = \sum_{i=0}^{n} a_i z^i$ with real positive coefficients satisfying $0<a_0 \leq a_1 \leq ... \leq a_n$ has all its zeros in $|z| \leq 1$. Various generalizations of this result are available in the literature. In this paper we put certain restrictions on the real and imaginary parts of the coefficients of a polynomial and find annular regions containing all the zeros of the polynomial. Our results generalize many results already known in the literature.

Keywords and Phrases: Zeros of polynomial, Eneström-Kakeya theorem.

2020 Mathematics Subject Classification: 12D10, 26C10 30C10, 30C15.

1. Introduction

Finding zeros of a polynomial is a long-standing classical problem [4-8]. Many results are available concerning the location of zeros of a polynomial of a complex variable with complex coefficients. It is an interesting area of research for engineers as well as mathematicians. Here we establish annular regions in which zeros of complex polynomial lie by placing various conditions on the real and imaginary parts of the complex coefficients of a given polynomial. Now we make use of the most classical result of Eneström-Kakeya Theorem namely Theorem A [1, 3].

Theorem A. [1, 3] *Let* $P(z)=\sum_{i=0}^{n}a_iz^i$ *be a polynomial of degree* n *such that* $0<a_0\leq a_1\leq ...\leq a_{n-1}\leq a_n$. *Then all the zeros of P(z) lie in* $|z|\leq 1$.

A. Joyal , G. Labelle and Q. I. Rahman [2] obtained the following generalization, by considering the coefficients to be real, instead of being only positive.

Theorem B. *Let* $P(z)=\sum_{i=0}^{n}a_iz^i$ *be a polynomial of degree* n *such that* $a_0\leq a_1\leq ...\leq a_{n-1}\leq a_n$. *Then all the zeros of P(z) lie in* $|z|\leq \frac{1}{|a_n|}\{a_n-a_0+|a_0|\}$.
By using the above results we establish the following generalizations.

Theorem 1. *Let* $P(z)=\sum_{i=0}^{n}a_iz^i$ *be a polynomial of degree* n *with complex coefficients. If* $Re(a_i)=\alpha_i,\ Im(a_i)=\beta_i$ *for* $i=0,1,2,...,n$ *and for some* $t\geq 0, l\leq n, m\leq n,\ \alpha_0\geq t\alpha_1\geq t^2\alpha_2\geq ...\geq t^{m-1}\alpha_{m-1}\geq t^m\alpha_m\leq t^{m+1}\alpha_{m+1}\leq ...\leq t^{n-1}\alpha_{n-1}\leq t^n\alpha_n$, *and* $\beta_0\geq t\beta_1\geq t^2\beta_2\geq ...\geq t^{l-1}\beta_{l-1}\geq t^l\beta_l\leq t^{l+1}\beta_{l+1}\leq ...\leq t^{n-1}\beta_{n-1}\leq t^n\beta_n$, *then all the zeros of P(z) lie in* $R_1\leq |z|\leq R_2$, *where* $R_1=min\{\frac{t^2|a_0|}{B_1},t\}$, $B_1=t(\alpha_0+\beta_0)-2(t^{m+1}\alpha_m+t^{l+1}\beta_l)+t^{n+1}(\alpha_n+\beta_n+|a_n|)$ *and* $R_2=max\{\frac{B_2}{|a_n|},\frac{1}{t}\}$,

$$B_2=|a_0|t^{n+1}+t^{n-1}(\alpha_0+\beta_0)+t(\alpha_n+\beta_n)-(1+t^2)(t^{n-m-1}\alpha_m+t^{n-l-1}\beta_l)$$
$$+(1-t^2)\Big[\sum_{i=1}^{m-1}t^{n-i-1}\alpha_i+\sum_{i=1}^{l-1}t^{n-i-1}\beta_i\Big]+(t^2-1)\Big[\sum_{i=m+1}^{n-1}t^{n-i-1}\alpha_i+\sum_{i=l+1}^{n-1}t^{n-i-1}\beta_i\Big].$$

Corollary 1. *Let* $P(z)=\sum_{i=0}^{n}a_iz^i$ *be a polynomial of degree* n *with complex coefficients. If* $Re(a_i)=\alpha_i,\ Im(a_i)=\beta_i$ *for* $i=0,1,2,...,n$ *and for some* $l\leq n, m\leq n$,

$$\alpha_0\geq\alpha_1\geq\alpha_2\geq ...\geq\alpha_{m-1}\geq\alpha_m\leq\alpha_{m+1}\leq ...\leq\alpha_{n-1}\leq\alpha_n \quad \text{and}$$

$$\beta_0\geq\beta_1\geq\beta_2\geq ...\geq\beta_{l-1}\geq\beta_l\leq\beta_{l+1}\leq ...\leq\beta_{n-1}\leq\beta_n,$$

then all the zeros of P(z) lie in

$$\frac{|a_0|}{(\alpha_0+\beta_0)-2(\alpha_m+\beta_l)+(\alpha_n+\beta_n+|a_n|)}\leq |z|\leq \frac{|a_0|+(\alpha_0+\beta_0)+(\alpha_n+\beta_n)-2(\alpha_m+\beta_l)}{|a_n|}.$$

Corollary 2. *Let* $P(z)=\sum_{i=0}^{n}a_iz^i$ *be a polynomial of degree* n *with real coefficients such that for some* $t\geq 0, m\leq n$ *and,* $a_0\geq ta_1\geq t^2a_2\geq ...\geq t^{m-1}a_{m-1}\geq t^ma_m\leq t^{m+1}a_{m+1}\leq ...\leq t^{n-1}a_{n-1}\leq t^na_n$. *Then all the zeros of P(z) lie in*

$R_3 \leq |z| \leq R_4$, *where* $R_3 = min\{\frac{t|a_0|}{B_3}, t\}$, $B_3 = a_0 - 2t^m a_m + t^n(a_n + |a_n|)$ *and* $R_4 = max\{\frac{B_4}{|a_n|}, \frac{1}{t}\}$, $B_4 = |a_0|t^{n+1} + t^{n-1}a_0 + ta_n - (1+t^2)t^{n-m-1}a_m + (1-t^2)\big[\sum_{i=1}^{m-1} t^{n-i-1}a_i\big] + (t^2-1)\big[\sum_{i=m+1}^{n-1} t^{n-i-1}a_i\big]$.

Remark 1. *By taking t=1 in Theorem 1, it reduces to Corollary 1.*

Remark 2. *By taking* $\beta = 0$ *in Theorem 1, it reduces to Corollary 2.*

Theorem 2. *Let* $P(z) = \sum_{i=0}^{n} a_i z^i$ *be a polynomial of degree n with complex coefficients. If* $Re(a_i) = \alpha_i$, $Im(a_i) = \beta_i$ *for* $i = 0, 1, 2, ..., n$ *and for some* $t \geq 0, l \leq n, m \leq n$, $\alpha_0 \geq t\alpha_1 \geq t^2\alpha_2 \geq ... \geq t^{m-1}\alpha_{m-1} \geq t^m\alpha_m \leq t^{m+1}\alpha_{m+1} \leq ... \leq t^{n-1}\alpha_{n-1} \leq t^n\alpha_n$ *and* $\beta_0 \leq t\beta_1 \leq t^2\beta_2 \leq ... \leq t^{l-1}\beta_{l-1} \leq t^l\beta_l \geq t^{l+1}\beta_{l+1} \geq ... \geq t^{n-1}\beta_{n-1} \geq t^n\beta_n$, *then all the zeros of P(z) lie in* $S_1 \leq |z| \leq S_2$,
where $S_1 = min\{\frac{t^2|a_0|}{C_1}, t\}$, $C_1 = t(\alpha_0 - \beta_0) - 2(t^{m+1}\alpha_m - t^{l+1}\beta_l) + t^{n+1}(\alpha_n - \beta_n + |a_n|)$ *and* $S_2 = max\{\frac{C_2}{|a_n|}, \frac{1}{t}\}$,

$$C_2 = |a_0|t^{n+1} + (\alpha_0 - \beta_0)t^{n-1} + (\alpha_n - \beta_n)t - (1+t^2)(t^{n-m-1}\alpha_m - t^{n-l-1}\beta_l)$$
$$+(1-t^2)\Big[\sum_{i=1}^{m-1} t^{n-i-1}\alpha_i - \sum_{i=1}^{l-1} t^{n-i-1}\beta_i\Big] + (t^2-1)\Big[\sum_{i=m+1}^{n-1} t^{n-i-1}\alpha_i - \sum_{i=l+1}^{n-1} t^{n-i-1}\beta_i\Big].$$

Corollary 3. *Let* $P(z) = \sum_{i=0}^{n} a_i z^i$ *be a polynomial of degree n with complex coefficients. If* $Re(a_i) = \alpha_i$, $Im(a_i) = \beta_i$ *for* $i = 0, 1, 2, ..., n$ *and for some* $l \leq n, m \leq n$,

$$\alpha_0 \geq \alpha_1 \geq \alpha_2 \geq ... \geq \alpha_{m-1} \geq \alpha_m \leq \alpha_{m+1} \leq ... \leq \alpha_{n-1} \leq \alpha_n \quad and$$

$$\beta_0 \leq \beta_1 \leq \beta_2 \leq ... \leq \beta_{l-1} \leq \beta_l \geq \beta_{l+1} \geq ... \geq \beta_{n-1} \geq \beta_n$$

then all the zeros of P(z) lie in

$$\frac{|a_0|}{(\alpha_0 - \beta_0) - 2(\alpha_m - \beta_l) + (\alpha_n - \beta_n + |a_n|)} \leq |z| \leq \frac{|a_0| + (\alpha_0 - \beta_0) + (\alpha_n - \beta_n) - 2(\alpha_m - \beta_l)}{|a_n|}.$$

Remark 3. *By taking t=1 in Theorem 2, it reduces to Corollary 3.*

By rearranging coefficients in Theorem 1 and Theorem 2 we get the following results.

Theorem 3. *Let* $P(z) = \sum_{i=0}^{n} a_i z^i$ *be a polynomial of degree n with complex coefficients. If* $Re(a_i) = \alpha_i$, $Im(a_i) = \beta_i$ *for* $i = 0, 1, 2, ..., n$ *and for some* $t \geq$

$0, l \leq n, m \leq n$, $\alpha_0 \leq t\alpha_1 \leq t^2\alpha_2 \leq ... \leq t^{m-1}\alpha_{m-1} \leq t^m\alpha_m \geq t^{m+1}\alpha_{m+1} \geq ... \geq t^{n-1}\alpha_{n-1} \geq t^n\alpha_n$ *and* $\beta_0 \geq t\beta_1 \geq t^2\beta_2 \geq ... \geq t^{l-1}\beta_{l-1} \geq t^l\beta_l \leq t^{l+1}\beta_{l+1} \leq ... \leq t^{n-1}\beta_{n-1} \leq t^n\beta_n$, *then all the zeros of P(z) lie in* $T_1 \leq |z| \leq T_2$,
where $T_1 = min\{\frac{t^2|a_0|}{D_1}, t\}$, $D_1 = (\beta_0-\alpha_0)t+2(t^{m+1}\alpha_m-t^{l+1}\beta_l)-t^{n+1}(\alpha_n-\beta_n-|a_n|)$ *and* $T_2 = max\{\frac{D_2}{|a_n|}, \frac{1}{t}\}$,

$$D_2 = |a_0|t^{n+1} + (\beta_0 - \alpha_0)t^{n-1} - (\alpha_n - \beta_n)t + (1+t^2)(t^{n-m-1}\alpha_m - t^{n-l-1}\beta_l)$$
$$+(t^2-1)\big[\sum_{i=1}^{m-1} t^{n-i-1}\alpha_i - \sum_{i=1}^{l-1} t^{n-i-1}\beta_i\big] + (1-t^2)\big[\sum_{i=m+1}^{n-1} t^{n-i-1}\alpha_i - \sum_{i=l+1}^{n-1} t^{n-i-1}\beta_i\big].$$

Corollary 4. *Let* $P(z) = \sum_{i=0}^{n} a_i z^i$ *be a polynomial of degree n with complex coefficients. If* $Re(a_i) = \alpha_i$, $Im(a_i) = \beta_i$ *for* $i = 0, 1, 2, ..., n$ *and for some* $l \leq n, m \leq n$,

$$\alpha_0 \leq \alpha_1 \leq \alpha_2 \leq ... \leq \alpha_{m-1} \leq \alpha_m \geq \alpha_{m+1} \geq ... \geq \alpha_{n-1} \geq \alpha_n \quad \text{and}$$

$$\beta_0 \geq \beta_1 \geq \beta_2 \geq ... \geq \beta_{l-1} \geq \beta_l \leq \beta_{l+1} \leq ... \leq \beta_{n-1} \leq \beta_n,$$

then all the zeros of P(z) lie in

$$\frac{|a_0|}{(\beta_0-\alpha_0)+2(\alpha_m-\beta_l)-(\alpha_n-\beta_n-|a_n|)} \leq |z| \leq \frac{|a_0|-(\alpha_0-\beta_0)-(\alpha_n-\beta_n)+2(\alpha_m-\beta_l)}{|a_n|}.$$

Corollary 5. *Let* $P(z) = \sum_{i=0}^{n} a_i z^i$ *be a polynomial of degree n with real coefficients such that for some* $t \geq 0, m \leq n$ *and* $a_0 \leq ta_1 \leq t^2a_2 \leq ... \leq t^{m-1}a_{m-1} \leq t^m a_m \geq t^{m+1}a_{m+1} \geq ... \geq t^{n-1}a_{n-1} \geq t^n a_n$. *Then all the zeros of P(z) lie in* $T_3 \leq |z| \leq T_4$, *where* $T_3 = min\{\frac{t|a_0|}{D_3}, t\}$, $D_3 = 2t^m a_m - a_0 + t^n(|a_n| - a_n)$ *and* $T_4 = max\{\frac{D_4}{|a_n|}, \frac{1}{t}\}$,

$$D_4 = |a_0|t^{n+1} - a_0t^{n-1} - ta_n + (1+t^2)t^{n-m-1}a_m$$
$$+ (t^2-1)\big[\sum_{i=1}^{m-1} t^{n-i-1}a_i\big] + (1-t^2)\big[\sum_{i=m+1}^{n-1} t^{n-i-1}a_i\big].$$

Remark 4. *By taking t=1 in Theorem 3, it reduces to Corollary 4.*

Remark 5. *By taking* $\beta = 0$ *in Theorem 3, it reduces to Corollary 5.*

2. Proof of the Theorems

Proof of Theorem 1. Let $P(z) = a_nz^n + a_{n-1}z^{n-1} + ... + a_1z + a_0$ be a polynomial of degree n with complex coefficients such that $Re(a_i) = \alpha_i,\ Im(a_i) = \beta_i$ for $i = 0, 1, 2, ..., n$.

Then consider the polynomial

$$\begin{aligned} Q(z) &= (t-z)P(z) \\ &= ta_0 + \sum_{i=1}^{n}(ta_i - a_{i-1})z^i - a_n z^{n+1} \\ &= ta_0 - F_1(z), \quad say \end{aligned}$$

On $|z| = t$,

$$\begin{aligned} |F_1(z)| \leq &\sum_{i=1}^{n}|ta_i - a_{i-1}|t^i + |a_n|t^{n+1} \\ \leq &\sum_{i=1}^{m}(\alpha_{i-1} - t\alpha_i)t^i + \sum_{i=m+1}^{n}(t\alpha_i - \alpha_{i-1})t^i + \sum_{i=1}^{l}(\beta_{i-1} - t\beta_i)t^i \\ &+ \sum_{i=l+1}^{n}(t\beta_i - \beta_{i-1})t^i + |a_n|t^{n+1} \\ = &\ t(\alpha_0 + \beta_0) - 2(t^{m+1}\alpha_m + t^{l+1}\beta_l) + t^{n+1}(\alpha_n + \beta_n + |a_n|) \\ = &\ B_1. \end{aligned}$$

Applying Schwarz lemma [9, 168] to $F_1(z)$, we get $|F_1(z)| \leq \frac{B_1|z|}{t} for\ |z| \leq t$, which implies $|Q(z)| = |ta_0 - F_1(z)| \geq t|a_0| - |F_1(z)| \geq t|a_0| - \frac{B_1|z|}{t}\ for\ |z| \leq t$.

Hence, if $|z| < R_1 = min\{\frac{t^2|a_0|}{B_1}, t\}$ then $Q(z) \neq 0$ and so $P(z) \neq 0$.

Next we show that $P(z) \neq 0$ for $|z| > R_2$.

For this, we again consider

$Q(z) = (t-z)P(z) = ta_0 + \sum_{i=1}^{n}(ta_i - a_{i-1})z^i - a_n z^{n+1} = F_2(z) - a_n z^{n+1}$ say.

Then $|z^n F_2(\frac{1}{z})| = |ta_0 z^n + \sum_{i=1}^{n}(ta_i - a_{i-1})z^{n-i}|$ and on $|z| = t$,

$$\begin{aligned} |z^n F_2(\frac{1}{z})| \leq &\ t|a_0|t^n + \sum_{i=1}^{n}|ta_i - a_{i-1}|t^{n-i} \\ \leq &\ |a_0|t^{n+1} + \sum_{i=1}^{m}(\alpha_{i-1} - t\alpha_i)t^{n-i} + \sum_{i=m+1}^{n}(t\alpha_i - \alpha_{i-1})t^{n-i} \\ &+ \sum_{i=1}^{l}(\beta_{i-1} - t\beta_i)t^{n-i} + \sum_{i=l+1}^{n}(t\beta_i - \beta_{i-1})t^{n-i} \end{aligned}$$

$$= |a_0|t^{n+1} + t^{n-1}(\alpha_0 + \beta_0) + t(\alpha_n + \beta_n) - (1+t^2)(t^{n-m-1}\alpha_m + t^{n-l-1}\beta_l)$$
$$+ (1-t^2)\Big[\sum_{i=1}^{m-1} t^{n-i-1}\alpha_i + \sum_{i=1}^{l-1} t^{n-i-1}\beta_i\Big] + (t^2-1)\Big[\sum_{i=m+1}^{n-1} t^{n-i-1}\alpha_i + \sum_{i=l+1}^{n-1} t^{n-i-1}\beta_i\Big]$$
$$= B_2.$$

Hence it follows by the Maximum Modulus theorem [9, p.165] that

$$|z^n F_2(\frac{1}{z})| \leq B_2 \; for \; |z| \leq t,$$

which implies

$$|F_2(z)| \leq B_2|z|^n \; for \; |z| \geq \frac{1}{t}.$$

From this it follows that

$$\begin{aligned} |Q(z)| &= |F_2(z) - a_n z^{n+1}| \\ &\geq |a_n||z|^{n+1} - B_2|z|^n \; for \; |z| \geq \frac{1}{t}, \\ &= |z|^n(|a_n||z| - B_2). \end{aligned}$$

Thus if $|z| > R_2 = max\{\frac{B_2}{|a_n|}, \frac{1}{t}\}$, then $Q(z) \neq 0$ and hence $P(z) \neq 0$.
This completes the proof of Theorem 1.

Proof of Theorem 2.

Let $P(z) = a_n z^n + a_{n-1}z^{n-1} + ... + a_1 z + a_0$ be a polynomial of degree n with complex coefficients such that $Re(a_i) = \alpha_i,\ Im(a_i) = \beta_i$ for $i = 0, 1, 2, ..., n$.
Then consider the polynomial

$$\begin{aligned} Q(z) &= (t-z)P(z) \\ &= ta_0 + \sum_{i=1}^{n}(ta_i - a_{i-1})z^i - a_n z^{n+1} = \; ta_0 - F_1(z) \; say. \end{aligned}$$

$On \; |z| = t,$

$$\begin{aligned} |F_1(z)| &\leq \sum_{i=1}^{n} |ta_i - a_{i-1}|t^i + |a_n|t^{n+1} \\ &\leq \sum_{i=1}^{m}(\alpha_{i-1} - t\alpha_i)t^i + \sum_{i=m+1}^{n}(t\alpha_i - \alpha_{i-1})t^i \\ &\quad + \sum_{i=1}^{l}(t\beta_i - \beta_{i-1})t^i + \sum_{i=l+1}^{n}(\beta_{i-1} - t\beta_i)t^i + |a_n|t^{n+1} \end{aligned}$$

$$= t(\alpha_0 - \beta_0) - 2(t^{m+1}\alpha_m - t^{l+1}\beta_l) + t^{n+1}(\alpha_n - \beta_n + |a_n|) = C_1.$$

Again applying Schwarz lemma [9, 168] to $F_1(z)$, we get $|F_1(z)| \leq \frac{C_1|z|}{t} for\ |z| \leq t$, which implies $|Q(z)| = |ta_0 - F_1(z)| \geq t|a_0| - |F_1(z)| \geq t|a_0| - \frac{C_1|z|}{t}\ for\ |z| \leq t$.

Hence, if $|z| < S_1 = min\{\frac{t^2|a_0|}{C_1}, t\}$ then $Q(z) \neq 0$ and so $P(z) \neq 0$.

Next we show that $P(z) \neq 0$ for $|z| > S_2$.

For this, we again consider
$Q(z) = (t - z)P(z) = ta_0 + \sum_{i=1}^{n}(ta_i - a_{i-1})z^i - a_n z^{n+1} = F_2(z) - a_n z^{n+1} \quad say.$
Then

$$|z^n F_2(\frac{1}{z})| = |ta_0 z^n + \sum_{i=1}^{n}(ta_i - a_{i-1})z^{n-i}|,$$

and on $|z| = t$,

$$|z^n F_2(\frac{1}{z})| \leq\ t|a_0|t^n + \sum_{i=1}^{n}|ta_i - a_{i-1}|t^{n-i}$$

$$\leq |a_0|t^{n+1} + \sum_{i=1}^{m}(\alpha_{i-1} - t\alpha_i)t^{n-i} + \sum_{i=m+1}^{n}(t\alpha_i - \alpha_{i-1})t^{n-i}$$

$$+ \sum_{i=1}^{l}(t\beta_i - \beta_{i-1})t^{n-i} + \sum_{i=l+1}^{n}(\beta_{i-1} - t\beta_i)t^{n-i}$$

$$= |a_0|t^{n+1} + (\alpha_0 - \beta_0)t^{n-1} + (\alpha_n - \beta_n)t - (1 + t^2)(t^{n-m-1}\alpha_m - t^{n-l-1}\beta_l)$$

$$+ (1 - t^2)\big[\sum_{i=1}^{m-1} t^{n-i-1}\alpha_i - \sum_{i=1}^{l-1} t^{n-i-1}\beta_i\big] + (t^2 - 1)\big[\sum_{i=m+1}^{n-1} t^{n-i-1}\alpha_i - \sum_{i=l+1}^{n-1} t^{n-i-1}\beta_i\big]$$

$$= C_2.$$

Hence it follows by the maximum modulus theorem [9, p.165] that

$$|z^n F_2(\frac{1}{z})| \leq C_2\ for\ |z| \leq t,$$

which implies

$$|F_2(z)| \leq C_2|z|^n\ for\ |z| \geq \frac{1}{t}.$$

From this we conclude that

$$\begin{aligned} |Q(z)| &= |F_2(z) - a_n z^{n+1}| \\ &\geq |a_n||z|^{n+1} - C_2|z|^n \ for \ |z| \geq \frac{1}{t}, \\ &= |z|^n(|a_n||z| - C_2). \end{aligned}$$

Thus if $|z|> S_2 = max\{\frac{C_2}{|a_n|}, \frac{1}{t}\}$, then $Q(z) \neq 0$ and hence $P(z) \neq 0$.

This completes the proof of Theorem 2.

Proof of Theorem 3 is similar to the Proof of Theorem 1 and Theorem 2.

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